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On the Computation of Optimal Monotone Mean-Variance Portfolios via Truncated Quadratic Utility[1]

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Abstract

We report a surprising link between optimal portfolios generated by a special type of variational preferences called divergence preferences (cf. [8]) and optimal portfolios generated by classical expected utility. As a special case we connect optimization of truncated quadratic utility (cf. [2]) to the optimal monotone mean-variance portfolios (cf. [9]), thus simplifying the computation of the latter.

Keywords: optimal portfolio, truncated quadratic utility, monotone mean-variance preferences, divergence preferences, HARA utility

JEL classification: G11, D81, C61

1 Introduction

This paper is motivated by two alternative recent attempts to deal with the non-monotonicity (in the sense of first order stochastic dominance) of quadratic utilities. The said non-monotonicity is a major drawback of these classical utility functions since the preference for more to less is a basic tenet of economic rationality.

The first approach, [2], uses expected truncated quadratic utility and leads to the so-called arbitrage-adjusted Sharpe ratio. The second, formulated in [9], modifies the variational form of mean-variance preferences. The two approaches are *prima facie* altogether different, with fundamental differences in the structure of preferences: in the first case we deal with expected utility, while in the second case we contend with divergence preferences, a special case of the variational preferences of [8]. Nevertheless, in this paper we show that there is an important and useful link between the optimal portfolios that the two approaches generate. This link is all the more interesting because variational preferences are closely related to convex risk measures (see [4] and [6]).

To illustrate this surprising link and its significance, let us step back and discuss the relation between the classical (non-monotone) expected quadratic utility and mean-variance preferences. Fix a probability space $(\Omega, \mathcal{F}, P)$ and define the quadratic utility $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = x - \frac{x^2}{2}. \tag{1}$$

If $Y \in L^2(P)$ is the value of an investment with zero initial outlay, its expected quadratic utility $F_q(Y) = E(f_q(Y))$ corresponds to

$$F_q(Y) = E(Y) - \frac{1}{2}E(Y^2), \tag{2}$$

while its mean-variance utility is

$$\Phi_q(Y) = E(Y) - \frac{1}{2}\mathrm{Var}(Y). \tag{3}$$

Formally, the link between the two utility functions is provided by the (not widely known) variational formula

$$\Phi_q(Y) = \inf_{Z \in L^2(P): E(Z)=1} E\left(ZY - f_q^*(Z)\right), \tag{4}$$

where $f_q^*(z) = -(1-z)^2/2$ is the Fenchel conjugate of f_q.

Regardless of whether one is aware of the link in (4), it is very well understood that optimal portfolios generated by the two criteria are closely related. This is easily checked by direct calculation. For an expected quadratic utility maximizer, the portfolio selection problem $\max_{\alpha \in \mathbb{R}} F_q(\alpha X)$ has a unique optimal portfolio $\hat{\alpha} = E(X)/E(X^2)$, with value

$$F_q(\hat{\alpha}X) = \frac{1}{2}\frac{\mathrm{SR}_X^2}{1 + \mathrm{SR}_X^2}, \tag{5}$$

where $\mathrm{SR}_X = E(X)/\sqrt{\mathrm{Var}(X)}$ is the so-called Sharpe ratio of the excess return X. The optimal portfolio $\hat{\alpha}$ and value $F_q(\hat{\alpha}X)$ are related to the optimal portfolio $\hat{\beta}$ and value $\Phi_q(\hat{\beta}X)$ of the corresponding mean-variance problem through the following relations

$$\hat{\beta} = \hat{\alpha}(1 + \mathrm{SR}_X^2) = \frac{\hat{\alpha}}{1 - 2F_q(\hat{\alpha}X)} \qquad \text{and} \qquad \Phi_q(\hat{\beta}X) = \frac{1}{2}\mathrm{SR}_X^2 = \frac{F_q(\hat{\alpha}X)}{1 - 2F_q(\hat{\alpha}X)}. \tag{6}$$

This is, in a nutshell, the classical relation between expected quadratic utility and mean-variance portfolio analysis. We will revisit these results at the end of Section 4. It is worth pointing out that when the random variable $\hat{\alpha}X$ is generalized to a stochastic integral $\hat{\alpha}\cdot X_T = \int_0^T \hat{\alpha}_t dX_t$, for suitably defined processes $\hat{\alpha}$ and X, then (6) represents a link between mean-square hedging and the dynamically efficient mean-variance portfolio (see [3, Lemma 5.1]).

The monotonization of quadratic preferences in [2] is based on a modification of the expected quadratic utility F_q. Specifically, [2] replaces the quadratic utility f_q in (1) with its monotone truncated version

$$f(x) = \begin{cases} x - \frac{1}{2}x^2 & \text{for } x \leq 1, \\ \frac{1}{2} & \text{for } x > 1. \end{cases} \tag{7}$$

Though in this case one no longer obtains simple expressions for the preference functional

$$F(Y) = E(f(Y)) \tag{8}$$

in terms of the first two moments of Y, [2] shows that solving the portfolio selection problem $\max_{\alpha\in\mathbb{R}^n} F(\alpha X)$ numerically is straightforward due to its low dimensionality and globally concave nature.[1]

The monotonization of mean-variance preferences proposed by [9] considers the mean-variance utility Φ_q. In particular, [9] replace Φ_q in (4) with its monotonization

$$\Phi(Y) = \inf_{Z\in L^2_+(P):E(Z)=1} E(ZY - f_q^*(Z)). \tag{9}$$

This is a simple specification of divergence preferences, which in general evaluate an investment Y through

$$\inf_{Z\in L^1_+(P):E(Z)=1} E(ZY - \phi(Z)),$$

where ϕ is a divergence kernel. Loosely speaking, divergence preferences consider (cautiously because of the inf) all possible probabilistic models Z for Y, penalized through their statistical distance $E(\phi(Z))$ from the reference model.[2] The monotone mean-variance preferences studied by [9] are the special case where $\phi = f_q^*$ is the Fenchel conjugate of the quadratic utility (1).

As argued in detail by [9], monotone mean-variance preferences are the monotone preferences that best approximate the original mean-variance preference functional (3). This gives monotone mean-variance preferences a solid decision theoretic underpinning as the proper monotone "correction" of mean-variance preferences. The portfolio selection problem $\max_{\beta\in\mathbb{R}^n}\Phi(\beta X)$ is, however, somewhat more complicated to solve than [2]'s problem $\max_{\alpha\in\mathbb{R}^n} F(\alpha X)$, though [9] manage to reformulate the optimization of Φ with n risky assets as a system of $n+1$ non-linear equations.

At first sight it is not at all obvious that there should be *any* correspondence between optimal portfolios generated by the monotone criteria (8) and (9). Nonetheless, it seems natural to enquire whether a result similar to (6) might hold for the optimal monotone portfolios. Exploratory numerical results based on [2] and [9] have shown that (6) indeed holds for the monotone quadratic portfolios provided that the Sharpe ratio in (6) is replaced by so-called arbitrage-adjusted Sharpe ratio introduced in [2]. This, of course, brings us no closer to understanding why such relationship should exist in the first place.

A first hint on the functional relation between the two approaches is the observation that when the Fenchel conjugate

$$f^*(z) = \begin{cases} -\frac{(1-z)^2}{2} & \text{for } z \geq 0 \\ -\infty & \text{for } z < 0 \end{cases}$$

[1]Here and in what follows, X is a vector of excess returns, and so α and β range over $\mathbb{R}^n$.

[2]See [8] for details. $E(\phi(Z))$ is the divergence of the probability ZdP from dP (see [7]).

of the truncated quadratic utility f in (7) is considered, then the preference functional Φ in (9) can be rewritten as

$$\Phi(Y)=\inf_{Z\in L_{+}^{2}(P):E(Z)=1}E(ZY-f^{*}(Z)). \tag{10}$$

The "monotonized" preference functionals F and Φ thus stand in a similar conjugate relation as the original F_q and Φ_q, that is, $F(Y)=E(f(Y))$ and $\Phi(Y)=\inf_{Z\in L_{+}^{2}(P):E(Z)=1}E(ZY-f^{*}(Z))$.

Our analysis builds on this intuition. Drawing on the recent work of [1] we show that the different approaches of [2] and [9] lead to closely related optimal portfolios. Our main result, Theorem 10, shows that *for a large class of objective functions* there is an explicit one-to-one relation among the solutions of the portfolio problems $\max_{\alpha\in\mathbb{R}^n}F(\alpha X)$ and $\max_{\beta\in\mathbb{R}^n}\Phi(\beta X)$, which allows to move back and forth between them. In particular, to find the solutions of $\max_{\beta\in\mathbb{R}^n}\Phi(\beta X)$ it is enough to solve the somewhat simpler problem $\max_{\alpha\in\mathbb{R}^n}F(\alpha X)$.

This surprising connection between these two portfolio problems is conceptually important because it allows to combine the decision theoretic appeal of the monotonization of [9] with the computational simplicity of that of [2]. We thus have a coherent picture, where the two different corrections of mean-variance preferences nicely complement each other.

2 Preliminaries

We collected in Appendix A few basic notions and results in Convex Analysis. For a concave function $f:\mathbb{R}\to[-\infty,\infty)$, we denote by $\operatorname{dom}_{+}f$ the largest open interval on which f is strictly increasing. It can be checked that $\operatorname{dom}_{+}f$ is a well defined set that coincides with the (possibly empty) set $\left\{x\in\operatorname{int}\operatorname{dom}f : f'_{+}(x)>0\right\}$.

Assumption 1 Throughout the paper, $f:\mathbb{R}\to[-\infty,\infty)$ is a proper, concave, increasing, and upper semicontinuous function, with $0\in\operatorname{dom}_{+}f$.

For all $Y\in L^{\infty}$, define

$$F(Y)=E\left(f(Y)\right), \tag{11}$$

and

$$\Phi(Y)=\inf_{Z\in L_{+}^{1}(P):E(Z)=1}E(ZY-f^{*}(Z)), \tag{12}$$

where $f^{*}:\mathbb{R}\to[-\infty,\infty)$ is the Fenchel concave conjugate of f, that is, $f^{*}(z)=\inf_{x\in\mathbb{R}}\{xz-f(x)\}$.

We already discussed these preference functionals in the Introduction. The functional F is a classical expected utility preference functional and, when f is the truncated quadratic utility (7), it is the preference functional used in [2]'s monotonization. The functional Φ is a divergence preference functional and, when $f=f_q$ it reduces to (9), on which the monotonization of [9] is based.

The next result collects the basic properties of the preference functional F.

Lemma 1 *The preference functional $F:L^{\infty}\to[-\infty,\infty)$ is proper, concave, increasing, and upper semicontinuous.*

In order to study the preference functional Φ we will restrict our attention to the following class of functions.

Definition 2 $\mathcal{H}$ denotes the set of functions f satisfying Assumption 1 and such that $f(0)=0$, $f'_{+}(0)\leq 1\leq f'_{-}(0)$, and there exist $x<0<y$ in $\operatorname{dom} f$ with $f'_{+}(x)>1$ and $1>f'_{+}(y)>0$.

For example, f belongs to $\mathcal{H}$ if it satisfies Assumption 1 and is twice continuously differentiable in a neighborhood of 0, with $f''(0) < f(0) = 0$ and $f'(0) = 1$.[3]

Lemma 3 *If $f \in \mathcal{H}$, then $1 \in \operatorname{int}\operatorname{dom} f^*$ and f^* attains its supremum at 1, with $f^*(1) = 0$.*

The next Theorem, essentially due to [1], provides the main link between Φ and F and establishes the basic properties of the functional Φ.

Theorem 4 *If $f \in \mathcal{H}$, then*

$$\Phi(Y) = \sup_{\eta\in\mathbb{R}}\{\eta + F(Y-\eta)\} = \max_{\eta\in[\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y]}\{\eta + F(Y-\eta)\}, \qquad \forall Y \in L^\infty(P).$$

Moreover, Φ is concave, increasing, normalized, translation invariant, finite, and Lipschitz.

3 Portfolio Selection Problems

Given a random vector $X \in L^\infty(\mathbb{R}^n)$ that represents the excess return of n securities, define the preference functionals $F_X, \Phi_X : \mathbb{R}^n \to [-\infty, \infty)$ over portfolios by setting

$$F_X(\alpha) = F(\alpha X) \qquad \text{and} \qquad \Phi_X(\beta) = \Phi(\beta X). \tag{13}$$

Provided the optimizers exist, we want to study the mutual relationship of the following portfolio selection problems

$$\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) \qquad \text{and} \qquad \max_{\beta\in\mathbb{R}^n} \Phi_X(\beta). \tag{14}$$

The first problem, $\max_{\alpha\in\mathbb{R}^n} F_X(\alpha)$, is a classical expected utility portfolio problem and, when f is the truncated quadratic (7), it is the problem considered by the monotonization of [2]. The second problem, $\max_{\beta\in\mathbb{R}^n} \Phi_X(\beta)$, is instead a novel, non expected utility, portfolio problem that uses divergence preferences. In particular, the monotone mean-variance preferences studied in [9] correspond to the preference functional Φ determined by the conjugate f^* of the quadratic $f(x) = x - x^2/2$.

Our aim is to determine the relations between the solutions and optimal values of the two problems in (14), that is,

$$\hat{\alpha}_X \in \arg\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) \qquad \text{and} \qquad \hat{F}_X = F_X(\hat{\alpha}_X), \tag{15}$$

$$\hat{\beta}_X \in \arg\max_{\beta\in\mathbb{R}^n} \Phi_X(\beta) \qquad \text{and} \qquad \hat{\Phi}_X = \Phi_X(\hat{\beta}_X). \tag{16}$$

We first establish the basic properties of F_X and Φ_X.

Lemma 5 *The function $F_X : \mathbb{R}^n \to [-\infty, \infty)$ is proper, concave, increasing, and upper semicontinuous. If $f \in \mathcal{H}$, then the function $\Phi_X : \mathbb{R}^n \to [-\infty, \infty)$ is real valued, concave, increasing, and Lipschitz.*

Next we define arbitrage free portfolios, which will play a key role in the solution of the two problems in (14).

Definition 6 *We say that $X \in L^\infty(\mathbb{R}^n)$ is arbitrage free if $\alpha X \geq 0$ implies $\alpha X = 0$ for all $\alpha \in \mathbb{R}^n$.*

[3] The condition $f''(0) < 0$ implies that f' is strictly decreasing on a neighbourhood of 0 and $f'(0) = 1$ guarantees that the neigborhood can be chosen so that f' is strictly positive on it. This yields the existence of $x < 0 < y$ such that $f'(x) > f'(0) = 1 > f'(y) > 0$.

The next two results establish the existence of solutions for problems (14). We begin by studying problem $\max_{\alpha\in\mathbb{R}^n} F_X(\alpha)$. We need some notation: set $f'(\infty) = \lim_{x\to\infty} f'_+(x)$ and, with a slight abuse of notation,

$$f'(-\infty) = \begin{cases} \lim_{x\to-\infty} f'_-(x) & \text{if } \mathrm{dom} f = \mathbb{R}, \\ \infty & \text{otherwise.} \end{cases}$$

Finally, set $\mathrm{sd}_+ f = \sup \mathrm{dom}_+ f$.

Theorem 7 *Suppose that $X \in L^\infty(\mathbb{R}^n)$ is arbitrage free and that $f'(\infty)/f'(-\infty) = 0$. Then, $\arg\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) \neq \emptyset$. Moreover, $f(0) \le \hat{F}_X < f(\mathrm{sd}_+ f)$.*

This result generalizes [2, Theorem 2], to which we refer for further relevant references. Turn now to the Φ_X optimization problem.

Theorem 8 *Suppose that $X \in L^\infty(\mathbb{R}^n)$ is arbitrage free. If f belongs to $\mathcal{H}$, with $f'(\infty) = 0$ and $f'(-\infty) = \infty$, then $\arg\max_{\beta\in\mathbb{R}^n} \Phi_X(\beta) \neq \emptyset$.*

Having just established their existence, we now proceed to examine optimal portfolios generated by divergence preferences. Along with Theorem 4, Theorem 8 implies the existence of $\hat{\beta}_X$ and $\hat{\eta}_X$ such that

$$\hat{\Phi}_X = \Phi(\hat{\beta}_X X) = \max_{\eta\in\mathbb{R},\beta\in\mathbb{R}^n} \{\eta + F(\beta X - \eta)\} = \max_{\eta\in\mathbb{R}} \{\eta + F(\hat{\beta}_X X - \eta)\} = \hat{\eta}_X + F(\hat{\beta}_X X - \hat{\eta}_X). \quad (17)$$

The quantity $\hat{\eta}_X$ may in general depend on $\hat{\beta}_X$ if the latter is not unique. In the next lemma we show that $-\hat{\eta}_X$ actually belongs to $\mathrm{dom}_+ f$, that is, to the strict monotonicity domain.

Lemma 9 *Suppose that $f \in \mathcal{H}$ and that $X \in L^\infty(\mathbb{R}^n)$ is arbitrage free. If $\hat{\beta}_X$ and $\hat{\eta}_X$ satisfy*

$$\hat{\Phi}_X = \hat{\eta}_X + F(\hat{\beta}_X X - \hat{\eta}_X),$$

then $-\hat{\eta}_X \in \mathrm{dom}_+ f$.

At this point we need to impose a specific structure on f in order to further proceed in our investigation of the relationships between the solution $\hat{\beta}_X$ and optimal value $\hat{\Phi}_X$ in (16) and the corresponding quantities $\hat{\alpha}_X$ and $\hat{F}_X$ in the expected utility maximization (15). Specifically, we will consider the rich class of HARA (hyperbolic absolute risk aversion) utility functions including logarithms, powers, and exponentials. We study this important class in the next section.

4 Main Result

Consider the following family of utility functions[4],

$$f_\gamma(x) = \begin{cases} \frac{(1+x/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } x \le -\gamma \\ \frac{\gamma}{\gamma-1} & \text{for } x > -\gamma \end{cases}, \qquad \gamma < 0, \text{ and} \quad (18)$$

$$f_\gamma(x) = \begin{cases} \frac{(1+x/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } x > -\gamma \\ -\infty & \text{for } x < -\gamma \end{cases}, \qquad 0 < \gamma \neq 1. \quad (19)$$

[4]Where not specified the value of f_γ at $-\gamma$ is defined as the unique value making f_γ upper semicontinuous.

Observe that in (18) and (19) one may compute pointwise limits as $\gamma \to \pm\infty$ and $\gamma \to 1$. We therefore define

$$f_1(x) = \begin{cases} \ln(1+x) & \text{for } x > -1 \\ -\infty & \text{for } x < -1 \end{cases}, \tag{20}$$

$$f_{\pm\infty}(x) = 1 - e^{-x}. \tag{21}$$

One easily verifies $f_\gamma \in \mathcal{H}$ for all $\gamma \in \bar{\mathbb{R}} \setminus \{0\}$.

Simple algebra shows that the Fenchel conjugate of f_γ is[5]

$$f_\gamma^*(z) = \begin{cases} \frac{-\gamma z^{1-1/\gamma} + (\gamma-1)z+1}{1/\gamma - 1} & \text{for } z > 0 \\ -\infty & \text{for } z < 0 \end{cases}, \qquad \gamma \in \mathbb{R}\backslash\{0,1\}, \tag{22}$$

with special cases

$$f_1^*(z) = \begin{cases} 1 - z + \ln z & \text{for } z > 0 \\ -\infty & \text{for } z < 0 \end{cases}, \tag{23}$$

$$f_{\pm\infty}^*(z) = \begin{cases} -z\ln z + z - 1 & \text{for } z > 0 \\ -\infty & \text{for } z < 0 \end{cases}. \tag{24}$$

It is important to observe that the Fenchel conjugates of monotone HARA utilities correspond to the most important family of divergence kernels. Specifically, $\left\{f_\gamma^*\right\}_{\gamma \neq 0}$ is the family of kernels of *power divergences* (see [7]), widely used in applications as they include the χ^2-divergence, the relative entropy, and the Hellinger distance.

The preference functionals (11) and (12) induced by f_γ are denoted by $F_\gamma, \Phi_\gamma : L^\infty \to \mathbb{R}$. Similarly, the optimal portfolios and values are denoted by $\hat{F}_{\gamma,X}$, $\hat{\alpha}_{\gamma,X}$, $\hat{\Phi}_{\gamma,X}$, and $\hat{\beta}_{\gamma,X}$. Using this notation, we can now state the main result of the paper. It shows that portfolio optimization with power divergence preferences can be solved in two stages, one of which involves solving optimal portfolio problem for expected HARA utility. Moreover, point (iii) establishes an explicit relationship between the optimal portfolios $\hat{\alpha}_{\gamma,X}$ and $\hat{\beta}_{\gamma,X}$, so that the knowledge of $\hat{\alpha}_{\gamma,X}$ is enough to determine $\hat{\beta}_{\gamma,X}$. As discussed in the Introduction, this is the main finding of the paper. Remarkably, point (iii) shows that $\hat{\alpha}_{\gamma,X}$ and $\hat{\beta}_{\gamma,X}$ feature the same mix of risky assets, though the leverage is different (and so the two portfolios can be very different).

Theorem 10 *Suppose $X \in L^\infty(\mathbb{R}^n)$ is arbitrage free. Then, for each $\gamma \in \bar{\mathbb{R}}\backslash\{0\}$ the maximizers $\hat{\alpha}_{\gamma,X}$ and $\hat{\beta}_{\gamma,X}$ exist. Moreover:*

(i) the maximizer $\hat{\eta}_{\gamma,X}$ is uniquely determined (i.e., it does not depend on $\hat{\beta}_{\gamma,X}$) and satisfies

$$\hat{\eta}_{\gamma,X} = \begin{cases} \gamma\left(1 - ((1/\gamma - 1)\,\hat{F}_{\gamma,X} + 1)^{1/\gamma}\right) & \text{for } \gamma \in \mathbb{R}\backslash\{0,1\} \\ 0 & \text{for } \gamma = 1 \\ -\ln(1 - \hat{F}_{\gamma,X}) & \text{for } \gamma = \pm\infty \end{cases}$$

(ii) the optimal values $\hat{F}_{\gamma,X}$ and $\hat{\Phi}_{\gamma,X}$ are in a one-to-one correspondence, as follows:

$$\hat{\Phi}_{\gamma,X} = \begin{cases} \frac{\gamma^2}{1-\gamma}\left((\hat{F}_{\gamma,X}\,(1/\gamma - 1) + 1)^{1/\gamma} - 1\right) & \text{for } \gamma \in \mathbb{R}\backslash\{0,1\} \\ \hat{F}_{\gamma,X} & \text{for } \gamma = 1 \\ -\ln(1 - \hat{F}_{\gamma,X}) & \text{for } \gamma = \pm\infty \end{cases}$$

[5] The value of f_γ^* at 0 is the unique value that makes f_γ^* upper semicontinuous.

(iii) the optimal portfolios for the two criteria are related as follows

$$\hat{\beta}_{\gamma,X} = \begin{cases} \hat{\alpha}_{\gamma,X}\left(\hat{F}_{\gamma,X}\,(1/\gamma - 1) + 1\right)^{1/\gamma} & \text{for } \gamma \in \mathbb{R}\backslash\{0,1\} \\ \hat{\alpha}_{\gamma,X} & \text{for } \gamma = 1 \\ \hat{\alpha}_{\gamma,X} & \text{for } \gamma = \pm\infty \end{cases}$$

where the equality is to be interpreted as equality of sets in $\mathbb{R}^n$.

Monotone mean-variance preferences correspond to $\gamma = -1$ and for them we readily recover

$$\begin{aligned} \widehat{\Phi}_{-1,X} &= \frac{\hat{F}_{-1,X}}{1 - 2\hat{F}_{-1,X}}, \\ \hat{\beta}_{-1,X} &= \hat{\alpha}_{-1,X}(1 - 2\hat{F}_{-1,X})^{-1}, \end{aligned}$$

which parallels the relationship between classical quadratic preferences in (6). [2] shows that the quantity $\sqrt{2\hat{F}_{-1,X}(1 - 2\hat{F}_{-1,X})^{-1}}$ can be interpreted as the arbitrage-adjusted Sharpe ratio (denoted by $\overline{\mathrm{SR}}$) of the optimal portfolio $\hat{\alpha}_{-1,X}X$. This allows us to write the result for the monotone mean-variance portfolio more intuitively as follows,

$$\begin{aligned} \widehat{\Phi}_{-1,X} &= \frac{1}{2}\overline{\mathrm{SR}}^2_{\hat{\alpha}_{-1,X}X}, \\ \hat{\beta}_{-1,X} &= \hat{\alpha}_{-1,X}(1 + \overline{\mathrm{SR}}^2_{\hat{\alpha}_{-1,X}X})^{-1}. \end{aligned}$$

The $n+1$ equations which characterize the optimal value $\hat{\beta}_{-1,X}$ in [9] are now readily seen to be the first order conditions of the optimization over η and β in (17) for the specific choice of $f(x) = f_{-1}(x) = -\frac{1}{2}(((1-x)^+)^2 - 1)$.

A Auxiliary Results in Convex Analysis

Here we summarize few useful results from Convex Analysis (we refer the interested reader to [11] for details). Given a closed and convex subset C of $\mathbb{R}^n$, intuitively its recession cone consists of all directions along which C is unbounded. Formally, the *recession cone* R_C of C is defined by

$$R_C = \{y \in \mathbb{R}^n : x + ty \in C \text{ for all } x \in C \text{ and all } t \geq 0\},$$

with the convention $R_\emptyset = \mathbb{R}^n$. The vectors in R_C are called *directions of recession.* It is easy to see that R_C is a closed convex cone.

The lineality space L_C is given by $R_C \cap R_{-C}$. In particular, L_C is a vector subspace, with

$$L_C = \{y \in \mathbb{R}^n : x + ty \in C \text{ for all } x \in C \text{ and all } t \in \mathbb{R}\}.$$

For a proper concave and upper semicontinuous function $f : \mathbb{R}^n \to [-\infty, \infty)$, the recession cone R_f and the lineality space L_f are given by $R_f = \bigcap_{\lambda\in\mathbb{R}} R_{(f\geq\lambda)}$ and $L_f = \bigcap_{\lambda\in\mathbb{R}} L_{(f\geq\lambda)}$. Clearly, $R_f \supseteq L_f$.

The following lemma gives a simple criterion to check whether a vector belongs to R_f or L_f (it actually holds in normed vector spaces; see [10]).

Lemma 11 *For a proper concave and upper semicontinuous function* $f : \mathbb{R}^n \to [-\infty, \infty)$, *the following statements are equivalent:*

(i) $y \in R_f$ *(resp.,* $y \in L_f$*);*

(iii) there is $x \in \mathbb{R}^n$ such that $\lim_{\lambda\to\infty} f(x+\lambda y) > -\infty$ (resp., $\lim_{\lambda\to\pm\infty} f(x+\lambda y) > -\infty$).

The following existence result summarizes several results in [11].

Theorem 12 *Let $f : \mathbb{R}^n \to [-\infty, \infty)$ be an upper semicontinuous proper concave function. Then $\arg\max_{x\in\mathbb{R}^n} f(x)$ is nonempty provided $R_f = L_f$.*

B Average Value at Risk

Consider a random variable $X \in L^\infty$ representing a financial position and its cumulative distribution function $F_X(x) = P(X \le x)$. Denote by q_X^- and q_X^+ the left- and right-continuous inverse of the cdf, respectively. That is,

$$\left.\begin{aligned} q_X^-(\lambda) &= \inf\{x \in \mathbb{R} : F_X(x) \ge \lambda\} \\ q_X^+(\lambda) &= \inf\{x \in \mathbb{R} : F_X(x) > \lambda\} \end{aligned}\right\} \text{ for } \lambda \in (0,1),$$

$$q_X^-(0) = q_X^+(0) = \text{ess } \inf X,$$

$$q_X^-(1) = q_X^+(1) = \text{ess } \sup X.$$

Following [5, A.14] the average value at risk (AVaR) at level $\lambda \in (0,1]$ is defined by

$$\text{AVaR}_X(\lambda) = -\frac{1}{\lambda}\int_0^\lambda q_X^+(z)dz.$$

For convenience we extend this definition to $\lambda = 0$ by setting

$$\text{AVaR}_X(0) = -\text{ess } \inf X = \lim_{\lambda\to 0} \text{AVaR}_X(\lambda). \tag{25}$$

The following result provides an alternative variational formula for AVaR.

Theorem 13 ([5, Lemma 4.46]) *Consider a random variable $X \in L^\infty$ and $\lambda \in (0,1]$. For any $q \in [q_X^-(\lambda), q_X^+(\lambda)]$*

$$\text{AVaR}_X(\lambda) = \frac{1}{\lambda}E((q-X)^+) - q = \frac{1}{\lambda}\inf_{\eta\in\mathbb{R}}(E((\eta - X)^+) - \lambda\eta).$$

The next proposition reports a novel link between recession directions of Φ, asymptotic slopes of f, and AVaR. This corrects, *inter alia*, a result of [1]. Their Theorem 2.2a) and Lemma 4.1 assert, in our notation,

$$\lim_{\lambda\to\infty} \Phi(\lambda Y)/\lambda = \operatorname{ess\,inf} Y, \qquad \forall f \in \mathcal{H}, \forall Y \in L^\infty,$$

which in fact only holds when f satisfies the conditions $f'(\infty) = 0$ and $f'(-\infty) = \infty$ appearing in equation (42).

Proposition 14 *If f belongs to $\mathcal{H}$, then*

$$\lim_{\lambda\to\infty} \frac{\Phi(\lambda Y)}{\lambda} = (1 - f'(\infty))\left(-\text{AVaR}_Y\left(\frac{1-f'(\infty)}{f'(-\infty) - f'(\infty)}\right)\right) + f'(\infty)E(Y), \qquad \forall Y \in L^\infty.$$

C Proofs and Related Analysis

Proof of Lemma 1. Consider the natural extension $\tilde{F}$ of F to L^1, defined by $\tilde{F}(Z) = E(f(Z))$. Next we show that $\tilde{F}$ is a well defined, concave, increasing, and proper function on L^1. In particular, $F : L^\infty \to [-\infty, \infty)$ is well defined, concave, increasing, and proper too. For all $c \in \mathbb{R}$ and $Z \in L^1$, $(f \circ Z)^{-1}([c,\infty))) = Z^{-1}\left(f^{-1}([c,\infty))\right)$. By the upper semicontinuity of f, $f^{-1}([c,\infty))$ is closed in $\mathbb{R}$. By the measurability of Z, $Z^{-1}\left(f^{-1}([c,\infty))\right)$ is a measurable set, and so $f \circ Z$ is a measurable function. Moreover, since f is proper, concave, and upper semicontinuous, there exist $a, b \in \mathbb{R}$ such that $f(x) \le ax + b$ for all $x \in \mathbb{R}$. Therefore $f \circ Z \le aZ + b$ and $E(f(Z)) \in [-\infty, \infty)$ for all $Z \in L^1$, i.e., $\tilde{F}$ is well defined. For all $\lambda \in (0,1)$ and $Z, W \in L^1$,

$$E(f(\lambda Z + (1-\lambda)W)) \ge E(\lambda f(Z) + (1-\lambda) f(W)) = \lambda E(f(Z)) + (1-\lambda) E(f(W)),$$

thus $\tilde{F}$ is concave. Monotonicity of f implies that, if $Z \ge W$ in L^1,[6] then $f(Z) \ge f(W)$ whence $E(f(Z)) \ge E(f(W))$, thus $\tilde{F}$ is increasing. Moreover, $E(f(x 1_\Omega)) = f(x) \in \mathbb{R}$ for all $x \in \text{dom } f$ implies that $\tilde{F}$ is proper since f is.

Next we show upper semicontinuity of F. Let $\{Y_n\}$ be a norm convergent sequence in L^∞ with limit $Y \in L^\infty$. Fix $m \ge 1$. There is $n_m \ge 1$ such that $Y_n \le_{a.s.} Y + 1/m$ for all $n \ge n_m$. Since f is increasing, this implies $F(Y_n) \le F(Y + 1/m)$ for all $n \ge n_m$, and so $\limsup_n F(Y_n) \le F(Y + 1/m)$. By the Levi Monotone Convergence Theorem, $\lim_{m\to\infty} F(Y + 1/m) = F(Y)$, whence $\limsup_n F(Y_n) \le F(Y)$.□

Under the assumptions of Lemma 1, we also have

$$\{Y \in L^\infty : \operatorname{ess\,inf}\ Y \in \text{dom } f\} \subseteq \text{dom } F \subseteq \{Y \in L^\infty : \operatorname{ess\,inf}\ Y \in \text{cl dom } f\}. \tag{26}$$

Notice that either dom $f = (d, \infty)$ with $d \in [-\infty, \infty)$ or dom $f = [d, \infty)$ with $d \in (-\infty, \infty)$. If dom $f = (-\infty, \infty)$, the three sets in (26) clearly coincide. Thus we assume $d \in \mathbb{R}$ and so cl dom $f = [d, \infty)$. If $Y \in L^\infty$ and $\operatorname{ess\,inf}\ Y \in \text{dom } f$, then $[\operatorname{ess\,inf}\ Y, \operatorname{ess\,sup}\ Y] \subseteq \text{dom } f$, $\operatorname{ess\,inf}\ Y \le_{a.s.} Y \le_{a.s.} \operatorname{ess\,sup}\ Y$ implies

$$-\infty < f(\operatorname{ess\,inf}\ Y) \le_{a.s.} f(Y) \le_{a.s.} f(\operatorname{ess\,sup}\ Y) < \infty.$$

Thus $f(Y) \in L^\infty$ and $F(Y) \in \mathbb{R}$. Now, assume per contra $Y \in \text{dom } F$ and $\operatorname{ess\,inf}\ Y \notin \text{cl dom } f$. Then

$$d > \operatorname{ess\,inf}\ Y = \max\{a \in \mathbb{R} : P(\{Y \ge a\}) = 1\}$$

and $P(\{Y \ge d\}) < 1$, thus $P(\{Y < d\}) > 0$. But $Y(\omega) < d$ implies $f(Y(\omega)) = -\infty$, thus $P(\{f(Y) = -\infty\}) > 0$ and $F(Y) = -\infty$, contradicting $Y \in \text{dom } F$.

The following example suggests why we cannot obtain a tighter result: Let $\Omega = (0,1)$ endowed with the Lebesgue σ-algebra and the Lebesgue measure. Set $Y_t(\omega) = \omega^t$ for all $\omega \in (0,1)$ and choose $f(x) = -x^{-1}$ if $x > 0$ and $-\infty$ otherwise. Then considering $Y_{1/2}$ and Y_1, it turns out that

$$\{Y \in L^\infty : \operatorname{ess\,inf}\ Y \in \text{dom } f\} \subset \text{dom } F \subset \{Y \in L^\infty : \operatorname{ess\,inf}\ Y \in \text{cl dom } f\}$$

since $Y_{1/2}$ belongs to the second but not to the first set, while Y_1 belongs to the third but not to the second one.

Proof of Lemma 3. Since $f : \mathbb{R} \to [-\infty, \infty)$ is proper, concave, and upper semicontinuous function, then the Young-Fenchel Theorem guarantees that for $x, y \in \mathbb{R}$,

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f^*(y) + f(x) = xy. \tag{27}$$

[6] With a common abuse, for measurable functions we will write $\ge$ to denote both the pointwise and the almost sure inequality. When we want to be more precise, we denote the latter by $\ge_{a.s.}$.

In particular,

$$\text{dom } f^* \supseteq \bigcup_{x \in \text{dom } f} \partial f(x) \supseteq \text{int dom } f^*. \tag{28}$$

Since $f'_+(0) \le 1 \le f'_-(0)$, then $1 \in \partial f(0)$, by (27), $0 \in \partial f^*(1)$, thus the maximum of f^* is attained at 1. By (27) again $f^*(1) = 0$. By (28), int $\bigcup_{x \in \text{dom } f} \partial f(x) = \text{int dom } f^*$ and $\bigcup_{x \in \text{dom } f} \partial f(x)$ is an interval. Since Definition 2 implies that there exist $z, w \in \mathbb{R}$ such that both $\partial f(z) \cap (0,1)$ and $\partial f(w) \cap (1, \infty)$ are nonempty, then $\bigcup_{x \in \text{dom } f} \partial f(x)$ contains an element strictly smaller than 1 and an element strictly greater than 1. Thus, $1 \in \text{int} \bigcup_{x \in \text{dom } f} \partial f(x)$ and $1 \in \text{int dom } f^*$. □

Proof of Theorem 4. Let $Y \in L^\infty(P)$. Set $\varphi(z) = -f^*(z)$, then $\varphi^*(x) = -f(-x)$ (i.e., $f(x) = -\varphi^*(-x)$). By Lemma 3, φ satisfies the assumptions of [1, Theorem 4.2], then, denoting by $\mathcal{P}$ the set of all probability measures that are absolutely continuous wrt P,

$$\begin{aligned}
\Phi(Y) &= \inf_{Z \in L^1_+(P): E(Z)=1} E(ZY - f^*(Z))\} = \inf_{Q \in \mathcal{P}} \left\{ \int Y dQ + \int -f^*\left(\frac{dQ}{dP}\right) dP \right\} \\
&= \inf_{Q \in \mathcal{P}} \left\{ \int Y dQ + \int \varphi\left(\frac{dQ}{dP}\right) dP \right\} = \sup_{\eta \in \mathbb{R}} \left\{ \eta - \int \varphi^*(\eta - X)\, dP \right\} \\
&= \sup_{\eta \in \mathbb{R}} \left\{ \eta + \int f(X - \eta)\, dP \right\}.
\end{aligned}$$

It is easy to check that Φ is concave, increasing, normalized, translation invariant, finite, and Lipschitz.

As to the second equality, set $d = \inf \text{dom } f$ and define $h_Y : \mathbb{R} \to [-\infty, \infty)$ by $h_Y(\eta) = \eta + E(f(Y - \eta))$. We have

$$\text{int dom } h_Y = (-\infty, \operatorname{ess\,inf} Y - d).$$

First observe that if $d = -\infty$ the equality is trivial. Assume $d \in \mathbb{R}$. If $\eta < \operatorname{ess\,inf} Y - d$, then $d < \operatorname{ess\,inf}(Y - \eta)$, by (26), $Y - \eta \in \text{dom } F$ and $\eta + E(f(Y - \eta))$ is finite. Thus $(-\infty, \operatorname{ess\,inf} Y - d)$ is an open subset of dom h_Y. Conversely, if $\eta > \operatorname{ess\,inf} Y - d$, then $\operatorname{ess\,inf}(Y - \eta) < d$, by (26), $Y - \eta \notin \text{dom } F$ and $\eta + E(f(Y - \eta)) = -\infty$. Thus, dom $h_Y \subseteq (-\infty, \operatorname{ess\,inf} Y - d]$ and int dom $h_Y \subseteq (-\infty, \operatorname{ess\,inf} Y - d)$.

It is easy to show that h_Y is concave. Set $\bar{\eta} = \operatorname{ess\,inf} Y - d$, to verify that h_Y is upper semicontinuous it is sufficient to check that $h_Y(\eta_n) \to h_Y(\bar{\eta})$ for each increasing sequence η_n converging to $\bar{\eta}$. This readily descends from the Monotone Convergence Theorem and the properties of f. In fact, $\eta_n \nearrow \bar{\eta}$ implies $Y - \eta_n \searrow Y - \bar{\eta}$ pointwise, then (by monotonicity and upper semicontinuity of f) $f(Y - \eta_n) \searrow f(Y - \bar{\eta})$, and it is pointwise dominated by a summable function. Thus, $E(f(Y - \eta_n)) \to E(f(Y - \bar{\eta}))$ and $\eta_n + E(f(Y - \eta_n)) \to \bar{\eta} + E(f(Y - \bar{\eta}))$.

Set $\zeta = \operatorname{ess\,sup} Y$. Since f is increasing, we have

$$h_Y(\eta) = \eta + E(f(Y - \eta)) \le \eta + E(f(\zeta - \eta)) = \eta + f(\zeta - \eta). \tag{29}$$

By Definition 2, there exist $x < 0 < y$ in dom f such that $f'_+(x) > 1$ and $1 > f'_+(y) > 0$. Denote by 1_A the indicator function of set A. For all $z \in \mathbb{R}$ we have

$$f(z) \le \big(f(x) + f'_+(x)(z - x)\big)\, 1_{(-\infty,0]}(z) + \big(f(y) + f'_+(y)(z - y)\big)\, 1_{(0,\infty)}(z).$$

Notice that $f(0) = 0$ implies that $f(z)/z$ is decreasing on its domain (and possibly constantly ∞ for z close to $-\infty$). Then,

$$\lim_{\lambda \to \infty} \frac{\lambda + f(-\lambda)}{\lambda} \le 1 - f'_+(x) < 0 \qquad \text{and} \qquad \lim_{\lambda \to \infty} \frac{-\lambda + f(\lambda)}{\lambda} \le f'_+(y) - 1 < 0.$$

Together with (29), this yields $\lim_{\eta\to\pm\infty} h_Y(\eta) \le \lim_{\eta\to\pm\infty} \eta + f(\zeta-\eta) = -\infty$. Hence, $h_Y : \mathbb{R} \to [-\infty,\infty)$ is coercive, and so it has a maximizer in $\mathbb{R}$. By [1, Proposition 2.1], the maximizer lies in $[\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y]$. □

Proof of Lemma 5. It is easy to see that F_X and Φ_X inherit the properties of F and Φ established in Lemma 1 and Theorem 4. We just check that Φ_X is Lipschitz. Let $\alpha, \beta \in \mathbb{R}^n$ and set $L = \max_{i=1,\ldots,n} \|X_i\|_\infty$. Since Φ is Lipschitz, say wlog with constant $M = 1$, we can write

$$\begin{aligned}|\Phi_X(\alpha) - \Phi_X(\beta)| &= |\Phi(\alpha X) - \Phi(\beta X)| \le \|\alpha X - \beta X\| = \operatorname{ess\,sup} |\alpha X - \beta X| \\ &= \operatorname{ess\,sup} \left|\sum_{i=1}^n (\alpha_i X_i - \beta_i X_i)\right| \le \operatorname{ess\,sup} \sum_{i=1}^n |\alpha_i - \beta_i|\,|X_i| \le L \sum_{i=1}^n |\alpha_i - \beta_i|,\end{aligned}$$

as desired. □

Proposition 15 *(i) For all $Y \in L^\infty$,*

$$\lim_{\lambda\to\infty} F(\lambda Y)/\lambda = -f'(-\infty)\, E(Y^-) + f'(\infty) E(Y^+), \tag{30}$$

under the convention $0 \cdot \infty = 0$.

(ii) If f satisfies

$$\frac{f'(\infty)}{f'(-\infty)} = 0, \tag{31}$$

then

$$\lim_{\lambda\to\infty} F(\lambda Y) = -\infty, \qquad \forall Y \notin L^\infty_+. \tag{32}$$

Proof. (i) Without loss of generality assume $f(0) = 0$. For every $x < 0 < y$ in $\operatorname{dom} f$ and arbitrary $\xi \in \partial f(x), \theta \in \partial f(y)$ we have

$$f(z) \le (f(x) + \xi(z-x))1_{(-\infty,0)}(z) + (f(y) + \theta(z-y))1_{(0,\infty)}(z),$$

where 1_A is the indicator function of set A and $z \in \mathbb{R}$. This implies

$$\frac{E(f(\lambda Y))}{\lambda} \le \frac{f(x) - \xi x}{\lambda} - \xi E(Y^-) + \frac{f(y) - \theta y}{\lambda} + \theta E(Y^+), \tag{33}$$

and consequently

$$\limsup_{\lambda\to\infty} F(\lambda Y)/\lambda \le -\xi E(Y^-) + \theta E(Y^+). \tag{34}$$

Let $y_n = n, \theta_n = f'_+(y_n)$. Depending on the $\operatorname{dom} f$,

$$\begin{aligned} &x_n = (1/d - 1/n)^{-1} \text{ and } \xi_n = f'_-(x_n) && \text{if } \operatorname{dom} f = (d,\infty), \\ &x_n = d \text{ and } \xi_n = f'_+(d) + n && \text{if } \operatorname{dom} f = [d,\infty). \end{aligned}$$

Note that by Assumption 1 $d < 0$. One easily verifies $x_n < 0 < y_n \in \operatorname{dom} f$, and $\xi_n \in \partial f(x_n), \theta_n \in \partial f(y_n)$ with $\xi_n \nearrow f'(-\infty)$ and $\theta_n \searrow f'(\infty)$. From (34)

$$\limsup_{\lambda\to\infty} F(\lambda Y)/\lambda \le \inf_{n\in\mathbb{N}} \{-\xi_n E(Y^-) + \theta_n E(Y^+)\} = -f'(-\infty)\, E\left(Y^-\right) + f'(\infty)\, E\left(Y^+\right). \tag{35}$$

We prove the converse inequalities case by case.

Case 1: $f'(-\infty) = \infty$ and $E(Y^-) > 0$. The rhs in (35) equals $-\infty$ which proves $\lim_{\lambda\to\infty} F(\lambda Y)/\lambda = -\infty$.

Case 2: $f'(-\infty) < \infty$. Recall $f(0) = 0$. For all $z \in \mathbb{R}$ we have,

$$f(z) \geq -f'(-\infty)z^- + f'(\infty)z^+,$$
$$\frac{F(\lambda Y)}{\lambda} \geq -f'(-\infty)E(Y^-) + f'(\infty)E(Y^+).$$

Case 3: $E(Y^-) = 0$. From $f(z) \geq f'(\infty)z$ for $z \geq 0$ we conclude $\frac{F(\lambda Y)}{\lambda} \geq f'(\infty)E(Y^+) = -f'(-\infty)E(Y^-) + f'(\infty)E(Y^+)$ due to the convention $0 \cdot \infty = 0$.

(ii) Consider $Y \notin L_+^\infty$. Then, $E(Y^-) > 0$. If $f'(-\infty) < \infty$, then (31) amounts to $f'(\infty) = 0$, and so (i) implies $\lim_{\lambda\to\infty} F(\lambda Y)/\lambda = -f'(-\infty)E(Y^-) < 0$. In turn, this implies $\lim_{\lambda\to\infty} F(\lambda Y) = -\infty$. On the other hand, if $f'(-\infty) = \infty$, then by (i) we have $\lim_{\lambda\to\infty} F(\lambda Y)/\lambda = -\infty$. Hence, also in this case $\lim_{\lambda\to\infty} F(\lambda Y) = -\infty$. □

By Lemma 11, condition (32) implies $R_F \subseteq L_+^\infty$. Condition (31) is also necessary for this inclusion, in the sense that, as the next example shows, when it is violated one can exhibit a finite probability space and $Y \notin L_+^\infty$ such that Y is a recession direction of F.

Example 16 If $f'(\infty)/f'(-\infty) > 0$, then $\infty > f'(-\infty) \geq f'(\infty) > 0$. With $f(0) = 0$ we have

$$f(z) \geq f'(\infty)z^+ - f'(-\infty)z^-.$$

Consider a finite probability space $\Omega = \{\omega_1, \omega_2\}$. Set $P(\{\omega_1\}) = p$ and take a random variable Y such that $Y(\omega_1) = -1$ and $Y(\omega_2) = 1$. If $p = f'(\infty)/(f'(\infty) + f'(-\infty))$, then the above implies

$$E(f(\lambda Y)) \geq f'(\infty)\lambda(1-p) - f'(-\infty)\lambda p = 0 > -\infty.$$

By Lemma 1, F is upper semicontinuous. By Lemma 11, $Y \in R_F$, and yet $Y \notin L_+^\infty$. □

Lemma 17 *If X is arbitrage free and $f'(\infty)/f'(-\infty) = 0$, then*

$$R_{F_X} = L_{F_X} = \{\alpha \in \mathbb{R}^n : \alpha X = 0\}. \tag{36}$$

Proof. By Lemma 11, $\alpha \in R_{F_X}$ only if $\lim_{\lambda\to\infty} F(\lambda\alpha X) > -\infty$. By Proposition 15, this implies $\alpha X \in L_+^\infty$. Since X is arbitrage free, we conclude $\alpha X = 0$. Hence, $R_{F_X} \subseteq \{\alpha \in \mathbb{R}^n : \alpha X = 0\}$. It remains to show that $\{\alpha \in \mathbb{R}^n : \alpha X = 0\} \subseteq L_{F_X}$. Let $\alpha \in \mathbb{R}^n$ be such that $\alpha X = 0$. Then, $\lim_{\lambda\to\infty} F(\lambda\alpha X) = \lim_{\lambda\to-\infty} F(\lambda\alpha X) = f(0) > -\infty$. Hence, $\alpha \in L_{F_X}$. □

Proof of Theorem 7. Since F_X is proper, concave, and upper semicontinuous, by Theorem 12 condition (36) implies $\arg\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) \neq \emptyset$. It remains to show that $f(0) \leq \hat{F}_X < f(\mathrm{sd}_+ f)$. Suppose $f(\mathrm{sd}_+ f) < \infty$, otherwise the statement is trivial. We have $f'_+(t + \mathrm{sd}_+ f) = 0$ for all $t > 0$. Then, $f(x) \leq f(t + \mathrm{sd}_+ f)$ for all $t > 0$ and all $x \in \mathbb{R}$, and so $f(\mathrm{sd}_+ f) = \max_{x\in\mathbb{R}} f(x)$. Since $0 \in \mathrm{dom}_+ f$, we have $f(0) \leq \hat{F}_X$ and $f(0) < \max_{x\in\mathbb{R}} f(x) = f(\mathrm{sd}_+ f)$.

From the absence of arbitrage, $\alpha X \notin L_+^\infty$, i.e., $P(\alpha X \leq 0) > 0$. Set $p = P(\alpha X \leq 0)$. Since $f(0) < f(\mathrm{sd}_+ f) < \infty$, we have

$$E(f(\alpha X)) \leq pf(0) + (1-p)f(\mathrm{sd}_+ f) < f(\mathrm{sd}_+ f), \qquad \forall \alpha \in \mathbb{R}^n,$$

and we conclude that $f(0) \leq \hat{F}_X < f(\mathrm{sd}_+ f)$. □

Proof of Proposition 14. Fix $\lambda > 0$. By Theorem 4 we have

$$\Phi(\lambda Y)/\lambda = \sup_{\eta\in\mathbb{R}}(\eta + F(\lambda Y - \eta))/\lambda = \sup_{\eta\in\mathbb{R}}\{\eta + F(\lambda(Y - \eta))/\lambda\}. \tag{37}$$

Consider the sequences $\{x_n\}, \{y_n\}, \{\xi_n\}, \{\theta_n\}$ defined in the proof of Proposition 15. Set $g(x, y, \xi, \theta) = f(x) + f(y) - \xi x - \theta y$. Equations (37) and (33) yield, for each $n \in \mathbb{N}$,

$$\Phi(\lambda Y)/\lambda \leq g(x_n, y_n, \xi_n, \theta_n)/\lambda + \sup_{\eta \in \mathbb{R}} \{\eta + \theta_n E((Y-\eta)^+) - \xi_n E((Y-\eta)^-)\},$$

and hence, for each $n \in \mathbb{N}$,

$$\begin{aligned}\limsup_{\lambda \to \infty} \Phi(\lambda Y)/\lambda &\leq \sup_{\eta \in \mathbb{R}} \left\{\eta + \theta_n E(Y-\eta) - (\xi_n - \theta_n) E((Y-\eta)^-)\right\} \\ &= -(1-\theta_n) \inf_{\eta \in \mathbb{R}} \left\{ \frac{\xi_n - \theta_n}{1-\theta_n} E((\eta - Y)^+) - \eta \right\} \\ &= -(1-\theta_n) \left(-\mathrm{AVaR}_Y \left(\frac{1-\theta_n}{\xi_n - \theta_n} \right) \right) + \theta_n E(Y), \end{aligned} \tag{38}$$

where the last equality follows from Theorem 13. On taking $\inf_{n \in \mathbb{N}}$ in (38) and in view of the continuity of AVaR_Y we obtain

$$\limsup_{\lambda \to \infty} \frac{\Phi(\lambda Y)}{\lambda} \leq (1 - f'(\infty)) \left(-\mathrm{AVaR}_Y \left(\frac{1 - f'(\infty)}{f'(-\infty) - f'(\infty)} \right) \right) + f'(\infty) E(Y). \tag{39}$$

We consider the converse inequalities case by case.

Case 1: $f'(-\infty) < \infty$. From (37), (35), and Theorem 13 we obtain

$$\begin{aligned}\Phi(\lambda Y)/\lambda &\geq \sup_{\eta \in \mathbb{R}} \{\eta + f'(\infty) E((Y-\eta)^+) - f'(-\infty) E((Y-\eta)^-)\} \\ &= (1 - f'(\infty)) \left(-\mathrm{AVaR}_Y \left(\frac{1 - f'(\infty)}{f'(-\infty) - f'(\infty)} \right) \right) + f'(\infty) E(Y). \end{aligned} \tag{40}$$

Combination of (39) and (40) completes the proof for $f'(-\infty) < \infty$.

Case 2: $f'(-\infty) = \infty$. On choosing $\eta = \operatorname{ess\,inf} Y$ equation (37) yields for all $\lambda > 0$

$$\Phi(\lambda Y)/\lambda \geq \operatorname{ess\,inf} Y + F(\lambda(Y - \operatorname{ess\,inf} Y))/\lambda.$$

From the proof of Case 3 in Proposition 15, we obtain

$$\begin{aligned}\liminf_{\lambda \to \infty} \Phi(\lambda Y)/\lambda &\geq \operatorname{ess\,inf} Y + f'(\infty) E((Y - \operatorname{ess\,inf} Y)^+) \\ &= (1 - f'(\infty)) \operatorname{ess\,inf} Y + f'(\infty) E(Y) \\ &= (1 - f'(\infty)) \left(-\mathrm{AVaR}_Y(0) \right) + f'(\infty) E(Y). \end{aligned} \tag{41}$$

Combination of (39) and (41) completes the proof for $f'(-\infty) = \infty$.

Lemma 18 *If f belongs to $\mathcal{H}$ and satisfies*

$$f'(\infty) = 0 \qquad \text{and} \qquad f'(-\infty) = \infty, \tag{42}$$

then

$$\lim_{\lambda \to \infty} \Phi(\lambda Y) = -\infty, \qquad \forall Y \notin L_+^\infty. \tag{43}$$

Proof Let $Y \notin L_+^\infty$, so that $\operatorname{ess\,inf} Y < 0$. By Proposition 14 and equation (25),

$$\begin{aligned}\lim_{\lambda \to \infty} \frac{\Phi(\lambda Y)}{\lambda} &= (1 - f'(\infty)) \left(-\mathrm{AVaR}_Y \left(\frac{1 - f'(\infty)}{f'(-\infty) - f'(\infty)} \right) \right) + f'(\infty) E(Y) \\ &= -\mathrm{AVaR}_Y(0) = \operatorname{ess\,inf} Y < 0. \end{aligned}$$

This implies $\lim_{\lambda\to\infty}\Phi(\lambda Y)=-\infty$. $\square$

By Lemma 11, condition (43) implies $R_\Phi \subseteq L_+^\infty$. Condition (42) is also necessary in the sense that when it is violated one can exhibit a finite probability space and $Y \notin L_+^\infty$ such that Y is a recession direction of Φ.

Example 19 Set $\lambda = \frac{1-f'(\infty)}{f'(-\infty)-f'(\infty)}$. Consider a finite probability space $\Omega=\{\omega_1,\omega_2\}$. Set $P(\{\omega_1\})=p$ and a random variable Y such that $Y(\omega_1)=-1$ and $Y(\omega_2)=y>0$. Clearly, $Y\notin L_+^\infty$. There are two cases to consider. (i) If $f'(-\infty)=\infty$, then $f'(\infty)>0$ and $\lambda=0$. Let $p=1/2$ and $y=2/f'(\infty)$. This yields,

$$-(1-f'(\infty))\mathrm{AVaR}_Y(\lambda)+f'(\infty)E(Y)=f'(\infty)/2>0.$$

(ii) If $f'(-\infty)<\infty$, we have $0<\lambda<1$, and we let $p=\lambda/2, y=1$. This yields

$$-(1-f'(\infty))\mathrm{AVaR}_Y(\lambda)+f'(\infty)E(Y)=(1-\lambda)f'(\infty)\geq 0.$$

In both cases $\liminf_{\lambda\to\infty}\Phi(\lambda Y)\geq 0$. By Theorem 4, Φ is finite valued and Lipschitz. By Lemma 11 $Y\in R_\Phi$, and yet $Y\notin L_+^\infty$. $\square$

Lemma 20 *If X is arbitrage free, $f'(\infty)=0$, and $f'(-\infty)=\infty$, then*

$$R_{\Phi_X}=L_{\Phi_X}=\{\alpha\in\mathbb{R}^n:\alpha X=0\}. \tag{44}$$

Proof. By Lemma 11, $\alpha\in R_{\Phi_X}$ only if $\lim_{\lambda\to\infty}\Phi(\lambda\alpha X)>-\infty$. By Proposition 15, this implies $\alpha X\in L_+^\infty$. Since X is arbitrage free, we conclude $\alpha X=0$. Hence, $R_{\Phi_X}\subseteq\{\alpha\in\mathbb{R}^n:\alpha X=0\}$. It remains to show that $\{\alpha\in\mathbb{R}^n:\alpha X=0\}\subseteq L_{\Phi_X}$. Let $\alpha\in\mathbb{R}^n$ be such that $\alpha X=0$. Then, $\lim_{\lambda\to\infty}\Phi(\lambda\alpha X)=\lim_{\lambda\to-\infty}\Phi(\lambda\alpha X)=\Phi(0)>-\infty$. Hence, $\alpha\in L_{\Phi_X}$. $\square$

Proof of Theorem 8. Since Φ_X is proper concave and upper semicontinuous, by Theorem 12 condition (44) implies $\arg\max_{\beta\in\mathbb{R}^n}\Phi_X(\beta)\neq\emptyset$. $\square$

Proof of Lemma 9. To ease notation we omit the X subscript in $\hat\beta$ and $\hat\eta$. First we show that $-\hat\eta<\mathrm{sd}_+f$. Suppose $\mathrm{sd}_+f<\infty$ (otherwise the inequality is trivially true). Suppose, per contra, that $-\hat\eta\geq\mathrm{sd}_+f$. Since $f(\mathrm{sd}_+f)=\max_{x\in\mathbb{R}}f(x)$, we have $f(\hat\beta X-\hat\eta)\leq f(\mathrm{sd}_+f)=f(0X-(-\mathrm{sd}_+f))$. Moreover, it is easy to check that $f\in\mathcal{H}$ implies $f(\mathrm{sd}_+f)<\mathrm{sd}_+f$. Then

$$\hat\eta+E\left(f(\hat\beta X-\hat\eta)\right)\leq-\mathrm{sd}_+f+f(\mathrm{sd}_+f)<0=0+E(f(0X-0)),$$

which contradicts the optimality of $\hat\beta,\hat\eta$. We conclude that $-\hat\eta<\mathrm{sd}_+f$.

Since $f\in\mathcal{H}$, we have $\operatorname{int}\operatorname{dom}f=(d,\infty)$, with $d\in[-\infty,0)$. Hence, $\mathrm{dom}_+f=(d,\mathrm{sd}_+f)$. It thus remains to show that $-\hat\eta>d$. Suppose $d>-\infty$ (otherwise the inequality is trivially true). By Theorem 4, $\operatorname{ess\,sup}(\hat\beta X)\geq\hat\eta\geq\operatorname{ess\,inf}(\hat\beta X)$. Moreover, $\operatorname{ess\,inf}(\hat\beta X)-\hat\eta\geq d$. For, otherwise $E(f(\hat\beta X-\hat\eta))=-\infty$.

Hence, we either have $\operatorname{ess\,inf}(\hat\beta X)<0$, and so $-\hat\eta>d$, or $\operatorname{ess\,inf}(\hat\beta X)=0$. In the latter case, the absence of arbitrage implies $\operatorname{ess\,sup}(\hat\beta X)=0$, and therefore $-\hat\eta=0>d$. In all cases, $-\hat\eta>d$. $\square$

Next we collect some elementary properties of the utility functions f_γ.

Lemma 21 *Consider the family of functions $\{f_\gamma\}_{\gamma\in\bar{\mathbb{R}}\setminus\{0\}}$. Then,*

(i) $f_\gamma\in\mathcal{H}$ for all γ, with $\mathrm{dom}f_\gamma=\mathbb{R}$ if $\gamma\in[-\infty,0)$;

(ii) $\mathrm{dom}_+ f_\gamma = \{x \in \mathbb{R} : 1 + x/\gamma > 0\}$ *for all* γ*, and*

$$f(\mathrm{sd}_+ f_\gamma) = \begin{cases} \frac{\gamma}{\gamma-1} & \text{for } \gamma \in [-\infty, 0) \cup (1, \infty] \\ \infty & \text{for } \gamma \in (0, 1] \end{cases}$$

(iii) $f''_\gamma(0) = -1$*,* $f'_\gamma(-\infty) = \infty$*, and* $f'_\gamma(\infty) = 0$ *for all* γ*;*

(iv) for $-\eta \in \mathrm{dom}_+ f_\gamma$*, we have*

$$f_\gamma(x - \eta) = \begin{cases} (1 - \eta/\gamma)^{1-\gamma} f_\gamma\left(x(1 - \eta/\gamma)^{-1}\right) + f_\gamma(-\eta) & \text{for } \gamma \in \mathbb{R}\backslash\{0\} \\ f_\gamma(x)(1 - f_\gamma(-\eta)) + f_\gamma(-\eta) & \text{for } \gamma = \pm\infty \end{cases}$$

Proof of Lemma 21. Parts (i)-(iii) are straightforward. In part (iv) it suffices to observe that one can express (18) equivalently as $f_\gamma(x) = \frac{((1+x/\gamma)^+)^{1-\gamma}-1}{1/\gamma-1}, \gamma < 0$, which makes the remaining computations straightforward.

Proof of Theorem 10. We prove all statements for $\gamma \in \mathbb{R}\backslash\{0, 1\}$. The proof for $\gamma = 1$ and $\gamma = \pm\infty$ follows along the same lines. First observe that the maximizers $\hat{\alpha}_{\gamma,X}$ and $\hat{\beta}_{\gamma,X}$ exist by Theorems 7 and 8.

(i) Fix a particular maximizer $\hat{\beta}$ and denote the corresponding optimal value of η in (17) by $\hat{\eta}$. Since $f_\gamma \in \mathcal{H}$, either $f_\gamma(\mathrm{sd}_+ f_\gamma) < \mathrm{sd}_+ f_\gamma$ or $\mathrm{sd}_+ f_\gamma = \infty$. By Lemma 9, $-\hat{\eta} \in \mathrm{dom}_+ f_\gamma$. Since $f_\gamma \in \mathcal{H}$, by Theorem 4

$$\hat{\Phi}_{\gamma,X} = \max_{\eta\in\mathbb{R}}\{\eta + \max_{\beta\in\mathbb{R}^n} F_\gamma(\beta X - \eta)\} = \max_{-\eta\in\mathrm{dom}_+ f_\gamma}\{\eta + \max_{\beta\in\mathbb{R}^n} E(f_\gamma(\beta X - \eta))\}.$$

Using Lemma 21,

$$\begin{aligned} \hat{\Phi}_{\gamma,X} &= \max_{\eta\in\mathbb{R}:1-\eta/\gamma>0}\{\eta + f_\gamma(-\eta) + (1 - \eta/\gamma)^{1-\gamma} \max_{\beta\in\mathbb{R}^n} E\left(f_\gamma\left(\beta X(1 - \eta/\gamma)^{-1}\right)\right)\} \\ &= \max_{\eta\in\mathbb{R}:1-\eta/\gamma>0}\{\eta + f_\gamma(-\eta) + (1 - \eta/\gamma)^{1-\gamma} \max_{\alpha\in\mathbb{R}^n} F_\gamma(\alpha X)\}. \end{aligned} \tag{45}$$

This proves that $\hat{\alpha} = (1 - \hat{\eta}/\gamma)^{-1} \hat{\beta}$ is a maximizer for $F_{\gamma,X}$ (otherwise the optimality of $\hat{\beta}, \hat{\eta}$ would be contradicted). We thus obtain

$$\hat{\Phi}_{\gamma,X} = \max_{\eta\in\mathbb{R}:1-\eta/\gamma>0}\{\eta + f_\gamma(-\eta) + (1 - \eta/\gamma)^{1-\gamma} \hat{F}_{\gamma,X}\}. \tag{46}$$

By Theorem 7, $0 \le \hat{F}_{\gamma,X} < f(\mathrm{sd}_+ f_\gamma)$, which implies

$$1 + \hat{F}_{\gamma,X}\,(1/\gamma - 1) > 0. \tag{47}$$

Define

$$h_{\gamma,\delta}(x) = \begin{cases} x + f_\gamma(-x) + (1 - x/\gamma)^{1-\gamma}\,\delta & \text{for } 1 - x/\gamma > 0 \\ -\infty & \text{for } 1 - x/\gamma < 0 \end{cases}$$

and define $h_{\gamma,\delta}(\gamma)$ to make the function upper semicontinuous. On $\mathrm{int\,dom}\, f$ we have

$$\begin{aligned} h'_{\gamma,\delta}(x) &= 1 - (1 - x/\gamma)^{-\gamma} \left(1 + \delta\,(1/\gamma - 1)\right), \\ h''_{\gamma,\delta}(x) &= -(1 - x/\gamma)^{-\gamma-1} \left(1 + \delta\,(1/\gamma - 1)\right). \end{aligned}$$

Therefore, $h_{\gamma,\delta}$ is strictly concave as long as

$$1 + \delta\,(1/\gamma - 1) > 0. \tag{48}$$

Assuming (48) holds one easily verifies that $h_{\gamma,\delta}$ has no directions of recession and therefore it possesses a unique maximizer. The point

$$\hat{x}_{\gamma,\delta} := \gamma(1 - ((1/\gamma - 1)\,\delta + 1)^{1/\gamma}),$$

satisfies $h'_{\gamma,\delta}(\hat{x}_{\gamma,\delta}) = 0$ and

$$1 - \hat{x}_{\gamma,\delta}/\gamma = ((1/\gamma - 1)\,\delta + 1)^{1/\gamma} > 0,$$

and so $\hat{x}_{\gamma,\delta}$ is an interior optimum of $h_{\gamma,\delta}$.

On taking $\delta = \widehat{F}_{\gamma,X}$ and in view of (47), equation (46) implies

$$\hat{\eta} = \gamma(1 - ((1/\gamma - 1)\,\hat{F}_{\gamma,X} + 1)^{1/\gamma}),$$

and therefore $\hat{\eta}$ does not depend on $\hat{\beta}$.

(ii) We have shown above

$$\hat{\eta}_{\gamma,X} = \gamma(1 - ((1/\gamma - 1)\,\hat{F}_{\gamma,X} + 1)^{1/\gamma}). \tag{49}$$

On substituting equation (49) into (46) we obtain the desired result.

(iii) Equation (45) implies that $\hat{\alpha}$ is a maximizer for $\widehat{F}_{\gamma,X}$ if and only if $\hat{\beta} = \hat{\alpha}\,(1 - \hat{\eta}_{\gamma,X}/\gamma)$ is a maximizer for $\widehat{\Phi}_{\gamma,X}$. Substitution from (49) yields the desired result. □

References

[1] Ben-Tal, A. and M. Teboulle, An old-new concept of convex risk measures: The optimized certainty equivalent, *Mathematical Finance*, 17, 449-476, 2007.

[2] Černý, A., Generalized Sharpe ratios and asset pricing in incomplete markets, *European Finance Review,* 7, 191-233, 2003.

[3] Černý, A. and J. Kallsen, Mean-variance hedging and optimal investment in Heston's model with correlation, *Mathematical Finance,* 18, 473-492, 2008.

[4] Föllmer, H. and A. Schied, Convex measures of risk and trading constraints, *Finance and Stochastics,* 6, 429-447, 2002.

[5] Föllmer, H. and A. Schied, *Stochastic finance: An introduction in discrete time*, (2nd ed.), Walter de Gruyter & Co, Berlin, 2004.

[6] Frittelli, M. and E. Rosazza Gianin, Putting order in risk measures, *Journal of Banking and Finance*, 26, 1473-1486, 2002.

[7] Liese, F. and I. Vajda, *Convex statistical distances*, Teubner, Leipzig, 1987.

[8] Maccheroni, F., M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica,* 74, 1447-1498, 2006.

[9] Maccheroni, F., M. Marinacci, A. Rustichini, and M. Taboga, Portfolio selection with monotone mean-variance preferences, *Mathematical Finance*, fothcoming.

[10] Rockafellar, R. T., Level sets and continuity of conjugate convex functions, *Transactions of the American Mathematical Society*, 123, 46-63, 1966.

[11] Rockafellar, R. T., *Convex analysis*, Princeton University Press, Princeton, 1970.